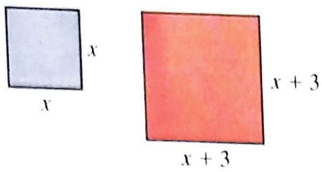
59. **Comparing Areas** Consider the two squares with sides x and $x + 3$ shown below.



a) Write a quadratic expression for the area of each square.

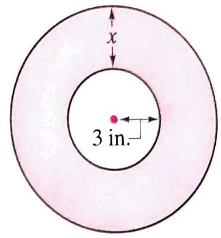
b) If the area of the blue square is 36 square inches, what is the length of each side of the square?

c) If the area of the blue square is 50 square inches, what is the length of each side of the square?

d) If the area of the red square is 81 square inches, what is the length of each side of the square?

e) If the area of the red square is 92 square inches, what is the length of each side of the square?

60. **Area** Consider the figure below. If the area shaded in pink is approximately 153.94 square inches, find x (to the nearest hundredth).



Challenge Problems

Use the square root property to solve for the indicated variable. Assume that all variables represent positive numbers. You may wish to review Section 2.6 before working these problems. List only the positive square root.

61. $A = s^2$, for s

62. $I = p^2r$, for p

63. $A = \pi r^2$, for r

64. $a^2 + b^2 = c^2$, for b

65. $I = \dfrac{k}{d^2}$, for d

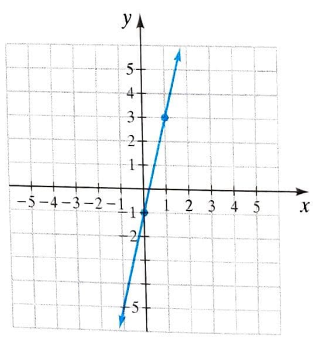
66. $A = p(1 + r)^2$, for r

Cumulative Review Exercises

[5.4] 67. Factor $6x^2 - 15x - 36$.

[6.5] 68. Simplify $\dfrac{5 - \frac{1}{y}}{6 - \frac{1}{y}}$.

[7.4] 69. Determine the equation of the line illustrated in the graph on the right.



See Exercise 69.

[9.4] 70. Simplify $\dfrac{\sqrt{135a^4b}}{\sqrt{3a^5b^7}}$.

10.2 Solving Quadratic Equations by Completing the Square

1 Write perfect square trinomials.

2 Solve quadratic equations by completing the square.

Quadratic equations that cannot be solved by factoring can be solved by completing the square or by the quadratic formula. In this section we focus on completing the square.

1 Write Perfect Square Trinomials

Perfect Square Trinomial

A **perfect square trinomial** is a trinomial that can be expressed as the square of a binomial.

Understanding Algebra

A trinomial that is a square of a binomial is called a *perfect square trinomial.*

Some examples follow.

Perfect Square Trinomials		Factors		Square of a Binomial
$x^2 + 6x + 9$	$=$	$(x + 3)(x + 3)$	$=$	$(x + 3)^2$
$x^2 - 6x + 9$	$=$	$(x - 3)(x - 3)$	$=$	$(x - 3)^2$
$x^2 + 10x + 25$	$=$	$(x + 5)(x + 5)$	$=$	$(x + 5)^2$
$x^2 - 10x + 25$	$=$	$(x - 5)(x - 5)$	$=$	$(x - 5)^2$

In a perfect square trinomial, *when the coefficient of the squared term is* 1, *the constant term is the square of one-half the coefficient of the x-term.*

Consider the perfect square trinomial $x^2 + 6x + 9$.

$$x^2 + 6 \quad + 9$$

6 is the coefficient of $6x$. 9 is the constant.

$$\left[\frac{1}{2}(6)\right]^2 = 3^2 = 9 \quad \text{Take } \frac{1}{2} \text{ of 6, and then square.}$$

Understanding Algebra

The constant term in a perfect square trinomial is always positive since you are squaring $\frac{1}{2}$ of the coefficient of the x-term.

Consider the perfect square trinomial $x^2 - 10x + 25$.

$$x^2 - 10\,x + 25$$

-10 is the coefficient of $-10x$. 25 is the constant.

$$\left[\frac{1}{2}(-10)\right]^2 = (-5)^2 = 25 \quad \text{Take } \frac{1}{2} \text{ of } -10\text{, and then square.}$$

Consider the expression $x^2 + 8x + \square$. Can you determine what number must be placed in the colored box to make the trinomial a perfect square trinomial? If you answered 16, you answered correctly.

$$x^2 + 8\,x + \square$$

$$\left[\frac{1}{2}(8)\right]^2 = 4^2 = 16 \quad \text{The constant is 16.}$$

The perfect square trinomial is $x^2 + 8x + 16$. Note that $x^2 + 8x + 16 = (x + 4)^2$.

Let's examine perfect square trinomials a little further.

Perfect Square Trinomial		Square of a Binomial
$x^2 + 6\,x + 9$	$=$	$(x + 3)^2$
$\frac{1}{2}(6)$	$=$	3
$x^2 - 10\,x + 25$	$=$	$(x - 5)^2$
$\frac{1}{2}(-10)$	$=$	-5

Note that when a perfect square trinomial is written as the square of a binomial *the constant in the binomial is one-half the value of the coefficient of the x-term in the perfect square trinomial.*

2 Solve Quadratic Equations by Completing the Square

In the following example, we solve a quadratic equation by **completing the square.** Several of the examples in this section *could* be solved by factoring, but we will solve them by completing the square to illustrate the procedure before solving more difficult problems.

EXAMPLE 1 Solve the equation $x^2 + 6x - 7 = 0$ by completing the square.

Solution First note that the squared term has a coefficient of 1. Next, to get the terms containing a variable by themselves on the left side of the equation, we add 7 to both sides of the equation.

$$x^2 + 6x - 7 = 0$$
$$x^2 + 6x = 7$$

Understanding Algebra

When completing the square, multiply $\frac{1}{2}$ by 6 to get $\frac{1}{2} \cdot 6 = 3$. After that, square the 3 to get $3^2 = 9$.

Now determine one-half the numerical coefficient of the x-term. In this example, the x-term is $6x$.

$$\frac{1}{2}(6) = 3$$

Square this number.

$$(3)^2 = (3)(3) = 9$$

Then add this product to both sides of the equation.

$$x^2 + 6x + 9 = 7 + 9$$

or

$$x^2 + 6x + 9 = 16$$

By following this procedure, we produce a perfect square trinomial on the left side of the equation. The expression $x^2 + 6x + 9$ is a perfect square trinomial that can be expressed as $(x + 3)^2$. Therefore,

$$x^2 + 6x + 9 = 16$$

can be written

$$(x + 3)^2 = 16$$

Now we use the square root property,

$$x + 3 = \pm\sqrt{16}$$
$$x + 3 = \pm 4$$

Finally, we solve for x by subtracting 3 from both sides of the equation.

$$x + 3 - 3 = -3 \pm 4$$
$$x = -3 \pm 4$$
$$x = -3 + 4 \quad \text{or} \quad x = -3 - 4$$
$$x = 1 \qquad\qquad x = -7$$

Thus, the solutions are 1 and -7. We check both solutions in the original equation.

Check $x = 1$

$$x^2 + 6x - 7 = 0$$
$$(1)^2 + 6(1) - 7 \stackrel{?}{=} 0$$
$$1 + 6 - 7 \stackrel{?}{=} 0$$
$$0 = 0 \quad \text{True}$$

$x = -7$

$$x^2 + 6x - 7 = 0$$
$$(-7)^2 + 6(-7) - 7 \stackrel{?}{=} 0$$
$$49 - 42 - 7 \stackrel{?}{=} 0$$
$$0 = 0 \quad \text{True}$$

Now Try Exercise 11

Now let's summarize the procedure.

To Solve a Quadratic Equation by Completing the Square

1. Use the multiplication (or division) property of equality if necessary to make the numerical coefficient of the squared term equal to 1.
2. Rewrite the equation with the constant by itself on the right side of the equation.
3. Take one-half the numerical coefficient of the first-degree term, square it, and add this quantity to both sides of the equation.
4. Replace the trinomial with its equivalent squared binomial.
5. Use the square root property.
6. Solve for the variable.
7. Check your answers in the *original* equation.

EXAMPLE 2 Solve the equation $x^2 - 10x + 21 = 0$ by completing the square.

Solution

$$x^2 - 10x + 21 = 0$$

$$x^2 - 10x = -21 \qquad \text{Step 2}$$

Take half the numerical coefficient of the x-term and then square it. You will add this product to both sides of the equation.

$$\frac{1}{2}(-10) = -5, \quad (-5)^2 = 25$$

Now add 25 to both sides of the equation.

$$x^2 - 10x + 25 = -21 + 25 \qquad \text{Step 3}$$

$$x^2 - 10x + 25 = 4$$

$$\text{or } (x - 5)^2 = 4 \qquad \text{Step 4}$$

$$x - 5 = \pm\sqrt{4} \qquad \text{Step 5}$$

$$x - 5 = \pm 2$$

$$x = 5 \pm 2 \qquad \text{Step 6}$$

$$x = 5 + 2 \quad \text{or} \quad x = 5 - 2$$

$$x = 7 \qquad\qquad x = 3$$

A check will show that the solutions are 7 and 3.

Now Try Exercise 13

Understanding Algebra

An important step in solving a quadratic equation by completing the square is to get the variable terms on the left side and the constant term on the right side of the equation. After that, complete the square with the terms on the left side.

EXAMPLE 3 Solve the equation $x^2 = 3x + 18$ by completing the square.

Solution Begin by subtracting $3x$ from both sides of the equation.

$$x^2 = 3x + 18$$

$$x^2 - 3x = 18 \qquad \text{Step 2}$$

Take half the numerical coefficient of the x-term, square it, and add this product to both sides of the equation.

$$\frac{1}{2}(-3) = -\frac{3}{2}, \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$x^2 - 3x + \frac{9}{4} = 18 + \frac{9}{4} \qquad \text{Step 3}$$

$$\left(x - \frac{3}{2}\right)^2 = 18 + \frac{9}{4} \qquad \text{Step 4}$$

Understanding Algebra

Remember the square of a negative number is a positive number. Thus,

$$\left(-\frac{3}{2}\right)^2 = \frac{9}{4} \text{ not } -\frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{72}{4} + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{81}{4}$$

$$x - \frac{3}{2} = \pm\sqrt{\frac{81}{4}} \qquad \text{Step 5}$$

$$x - \frac{3}{2} = \pm\frac{9}{2}$$

$$x = \frac{3}{2} \pm \frac{9}{2} \qquad \text{Step 6}$$

$$x = \frac{3}{2} + \frac{9}{2} \quad \text{or} \quad x = \frac{3}{2} - \frac{9}{2}$$

$$x = \frac{12}{2} = 6 \qquad x = -\frac{6}{2} = -3$$

The solutions are 6 and -3.

Now Try Exercise 21

In the following examples we will not show some of the intermediate steps.

EXAMPLE 4 Solve the equation $x^2 - 16x + 12 = 0$ by completing the square.

Solution

$$x^2 - 16x + 12 = 0$$

$$x^2 - 16x = -12 \qquad \text{Step 2}$$

$$x^2 - 16x + 64 = -12 + 64 \qquad \text{Step 3}$$

$$(x - 8)^2 = 52 \qquad \text{Step 4}$$

$$x - 8 = \pm\sqrt{52} \qquad \text{Step 5}$$

$$x - 8 = \pm\sqrt{4}\sqrt{13}$$

$$x - 8 = \pm 2\sqrt{13}$$

$$x = 8 \pm 2\sqrt{13} \qquad \text{Step 6}$$

The solutions are $8 + 2\sqrt{13}$ and $8 - 2\sqrt{13}$.

Now Try Exercise 29

EXAMPLE 5 Solve the equation $5z^2 - 25z + 10 = 0$ by completing the square.

Solution Since the coefficient of the squared term is 5, we multiply both sides of the equation by $\frac{1}{5}$ (or divide every term by 5) to make the coefficient equal to 1.

$$5z^2 - 25z + 10 = 0$$

$$\frac{1}{5}(5z^2 - 25z + 10) = \frac{1}{5}(0) \qquad \text{Step 1}$$

$$z^2 - 5z + 2 = 0$$

Now we proceed as in earlier examples.

$$z^2 - 5z = -2 \qquad \text{Step 2}$$

$$z^2 - 5z + \frac{25}{4} = -2 + \frac{25}{4} \qquad \text{Step 3}$$

$$\left(z - \frac{5}{2}\right)^2 = -\frac{8}{4} + \frac{25}{4} \qquad \text{Step 4}$$

$$\left(z - \frac{5}{2}\right)^2 = \frac{17}{4}$$

$$z - \frac{5}{2} = \pm\sqrt{\frac{17}{4}} \qquad \text{Step 5}$$

$$z - \frac{5}{2} = \pm\frac{\sqrt{17}}{2}$$

$$z = \frac{5}{2} \pm \frac{\sqrt{17}}{2} \qquad \text{Step 6}$$

$$z = \frac{5}{2} + \frac{\sqrt{17}}{2} \quad \text{or} \quad z = \frac{5}{2} - \frac{\sqrt{17}}{2}$$

$$z = \frac{5 + \sqrt{17}}{2} \qquad z = \frac{5 - \sqrt{17}}{2}$$

The solutions are $\frac{5 + \sqrt{17}}{2}$ and $\frac{5 - \sqrt{17}}{2}$.

Now Try Exercise 33

EXERCISE SET 10.2

MathXL® MyMathLab

Warm-Up Exercises

Fill in the blanks with the appropriate word, phrase, or symbol(s) from the following list.

9	perfect square	add 11	6	$x + 5$	36
$\frac{9}{2}$	add -11	$\frac{81}{4}$	$x - 5$	square of a binomial	

1. A __________ trinomial is a trinomial that can be expressed as the square of a binomial.
2. The square of the binomial __________ is the trinomial $x^2 - 10x + 25$.
3. $x^2 - 7x + 2$ is not a perfect square trinomial because it cannot be written as the __________.
4. In a perfect square trinomial with a leading coefficient of 1, if the coefficient of the x-term is 12, the constant term is __________.
5. In a perfect square trinomial with a leading coefficient of 1, if the coefficient of the x-term is 9, the constant term is __________.
6. The first step in solving the equation $x^2 + 4x - 11 = 0$ by completing the square is to __________ to both sides.

Practice the Skills

Solve by completing the square.

7. $x^2 + 7x + 10 = 0$
8. $x^2 + 9x + 18 = 0$
9. $x^2 - 8x + 7 = 0$
10. $r^2 - 2r - 35 = 0$
11. $x^2 + 13x + 12 = 0$
12. $x^2 + 13x + 36 = 0$
13. $z^2 - 6z + 8 = 0$
14. $m^2 - 12m + 35 = 0$
15. $n^2 = -6n - 9$
16. $k^2 = 14k - 49$
17. $x^2 = 2x + 15$
18. $x^2 = -5x - 6$
19. $x^2 + 10x + 24 = 0$
20. $-40 = n^2 + 13n$
21. $x^2 = 10x - 16$
22. $x^2 = 2x + 35$
23. $-60 = -p^2 + 4p$
24. $-x^2 - 6x + 55 = 0$
25. $z^2 - 4z = -2$
26. $z^2 + 2z = 10$
27. $w^2 + 6w = -3$
28. $g^2 - 2g = 9$
29. $m^2 + 7m + 2 = 0$
30. $x^2 + 3x - 3 = 0$
31. $3x^2 + 6x - 9 = 0$
32. $2x^2 + 2x - 24 = 0$
33. $2x^2 + 18x + 4 = 0$
34. $3x^2 + 21x - 12 = 0$
35. $3h^2 - 15h = 18$
36. $4x^2 = -28x + 32$
37. $3x^2 - 11x - 4 = 0$
38. $3x^2 - 8x + 4 = 0$
39. $9t^2 + 6t = 6$
40. $6x^2 - 3x = 15$
41. $2x^2 - 16x = 0$
42. $6x^2 - 42x = 0$
43. $3x^2 = 27x$
44. $7x^2 = 28x$

Problem Solving

45. a) Write a perfect square trinomial that has a term of $10x$.
    b) Explain how you constructed your perfect square trinomial.

46. a) Write a perfect square trinomial that has a term of $-16x$.
    b) Explain how you constructed your perfect square trinomial.

47. **Numbers** When 3 times a number is added to the square of a number, the sum is 4. Find the number(s).

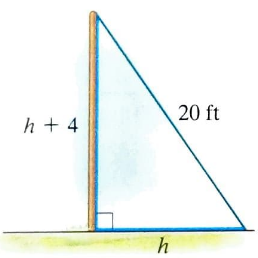
48. **Numbers** When 5 times a number is subtracted from 2 times the square of a number, the difference is 12. Find the number(s).

49. **Numbers** If the square of 3 more than a number is 9, find the number(s).

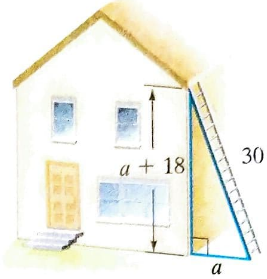
50. **Numbers** If the square of 2 less than an integer is 16, find the number(s).

51. **Numbers** The product of two positive numbers is 21. Find the two numbers if the larger is 4 greater than the smaller.

52. **Supporting a Pole** A guy wire 20 feet long is supporting a pole as shown in the figure. Determine the height of the pole.



53. **Ladder Leaning on a House** A 30-foot ladder is leaning against a house, as shown. Determine the vertical distance from the ground to where the ladder rests on the house.



54. **Sum of Integers** The sum of the first n integers, s, can be found by the formula $s = \frac{n^2 + n}{2}$. Find the value of n if the sum is 28.

55. **Height of an Object** When an object is thrown straight up from Earth with an initial velocity of 128 feet per second, its height above the ground, s, in feet, in t seconds is given by the formula $s = -16t^2 + 128t$. How long will it take the object to reach a height of 240 feet? (Therefore, $s = 240$.)

56. **Height of an Object** Repeat Exercise 55 for a height of 112 feet.

Challenge Problems

57. Fill in the shaded area to make a perfect square trinomial.

$$x^2 \quad\quad + 81$$

58. Fill in the shaded area to make a perfect square trinomial.

$$x^2 \quad\quad + \frac{9}{100}$$

59. a) Solve the equation $x^2 - 14x - 1 = 0$ by completing the square.
    b) Check your solution (it will not be a rational number) by substituting the value(s) you obtained in part **a)** for each x in the equation in part **a)**.

60. a) Solve the equation $x^2 + 3x - 7 = 0$ by completing the square.
    b) Check your solution (it will not be a rational number) by substituting the value(s) you obtained in part **a)** for each x in the equation in part **a)**.

Solve by completing the square.

61. $x^2 + \frac{3}{5}x - \frac{1}{2} = 0$

62. $x^2 - \frac{2}{3}x - \frac{1}{5} = 0$

63. $3x^2 + \frac{1}{2}x = 4$

64. $0.1x^2 + 0.2x - 0.54 = 0$

65. $-5.26x^2 + 7.89x + 15.78 = 0$

Cumulative Review Exercises

[6.4] 66. Simplify $\frac{x^2}{x^2 - x - 6} - \frac{x - 2}{x - 3}$.

[7.4] 67. Explain how you can determine whether two equations represent parallel lines without graphing the equations.

[8.2, 8.3] 68. Solve the following system of equations.

$$2x + 3y = 6$$
$$-x + 4y = 19$$

[9.5] 69. Solve the equation $\sqrt{2x + 3} = 2x - 3$.