10.4 Graphing Quadratic Equations

1 Graph quadratic equations in two variables.

2 Find the coordinates of the vertex of a parabola.

3 Use symmetry to graph quadratic equations.

4 Find the intercepts of the graph of a quadratic equation.

1 Graph Quadratic Equations in Two Variables

Helpful Hint

Study Tip

In this section we will graph quadratic equations. Quadratic equations can be graphed by plotting points, as we did in Chapter 7 when we graphed linear equations. However, there are certain things we can do to make graphing quadratic equations easier. These include finding the vertex of the graph, using symmetry to draw the graph, and finding the intercepts of the graph.

In Section 7.2 we learned how to graph linear equations. In this section we graph quadratic equations of the form

$$y = ax^2 + bx + c, \quad a \neq 0$$

The graph of every quadratic equation is a **parabola**. The graph of $y = ax^2 + bx + c$ will have one of the shapes indicated in **Figure 10.3**.

FIGURE 10.3

The **vertex** is the lowest point on a parabola that opens upward or the highest point on a parabola that opens downward (**Fig. 10.4**). Graphs of quadratic equations of the form $y = ax^2 + bx + c$ have **symmetry** about a line through the vertex. This means that if we fold the paper along this imaginary line, called the **axis of symmetry**, the right and left sides of the graph match up.

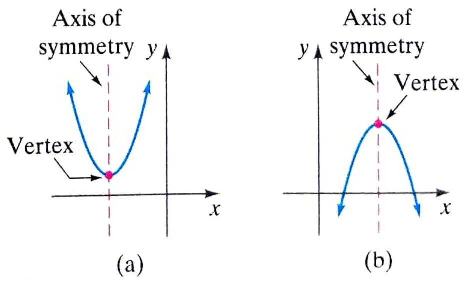


FIGURE 10.4

One method to graph a quadratic equation is to plot it point by point. Select values for x and determine the corresponding values for y. Then plot the ordered pairs.

EXAMPLE 1 Graph $y = x^2$.

Solution Since $a = 1$, which is positive, this parabola opens upward.

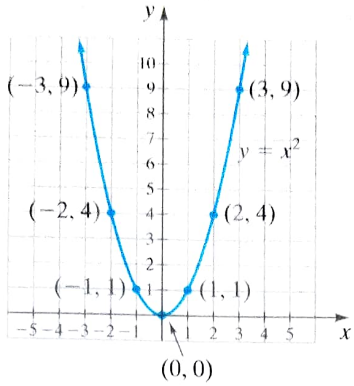


FIGURE 10.5

	$y = x^2$
Let $x = 3$,	$y = (3)^2 = 9$
Let $x = 2$,	$y = (2)^2 = 4$
Let $x = 1$,	$y = (1)^2 = 1$
Let $x = 0$,	$y = (0)^2 = 0$
Let $x = -1$,	$y = (-1)^2 = 1$
Let $x = -2$,	$y = (-2)^2 = 4$
Let $x = -3$,	$y = (-3)^2 = 9$

x	y
3	9
2	4
1	1
0	0
−1	1
−2	4
−3	9

Connect the points with a smooth curve (**Fig. 10.5**). Note how the graph is symmetric about the line $x = 0$ (or the y-axis).

Now Try Exercise 21

EXAMPLE 2 Graph $y = -2x^2 + 4x + 6$.

Solution Since $a = -2$, which is negative, this parabola opens downward.

	$y = -2x^2 + 4x + 6$
Let $x = 5$,	$y = -2(5)^2 + 4(5) + 6 = -24$
Let $x = 4$,	$y = -2(4)^2 + 4(4) + 6 = -10$
Let $x = 3$,	$y = -2(3)^2 + 4(3) + 6 = 0$
Let $x = 2$,	$y = -2(2)^2 + 4(2) + 6 = 6$
Let $x = 1$,	$y = -2(1)^2 + 4(1) + 6 = 8$
Let $x = 0$,	$y = -2(0)^2 + 4(0) + 6 = 6$
Let $x = -1$,	$y = -2(-1)^2 + 4(-1) + 6 = 0$
Let $x = -2$,	$y = -2(-2)^2 + 4(-2) + 6 = -10$
Let $x = -3$,	$y = -2(-3)^2 + 4(-3) + 6 = -24$

x	y
5	−24
4	−10
3	0
2	6
1	8
0	6
−1	0
−2	−10
−3	−24

The graph of $y = -2x^2 + 4x + 6$ is shown in **Figure 10.6**. The graph is symmetric about the line $x = 1$ (which is dashed since it is not part of the graph).

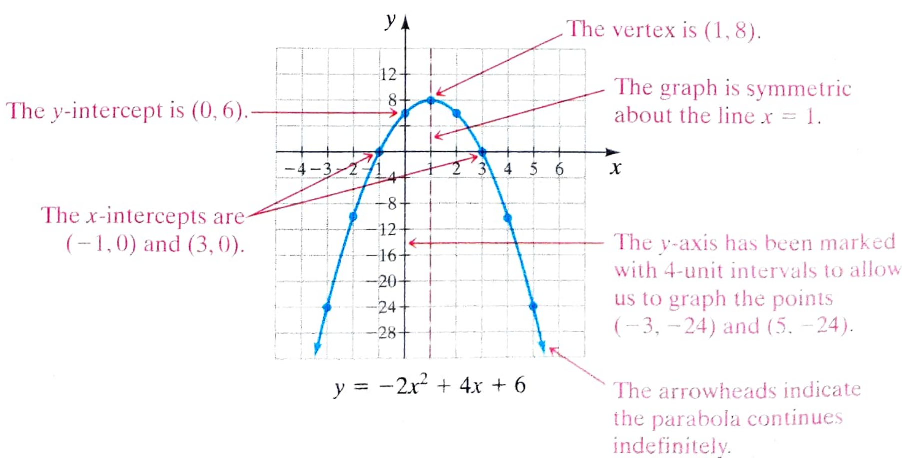


FIGURE 10.6

Now Try Exercise 23

2 Find the Coordinates of the Vertex of a Parabola

When the location of the vertex is known, it is easier to decide which values to use for x when plotting points. For a quadratic equation in the form $y = ax^2 + bx + c$, both the axis of symmetry and the x-coordinate of the vertex can be found by using the following formula.

Understanding Algebra

The vertex of a parabola is always on the axis of symmetry.

Axis of Symmetry and x-Coordinate of the Vertex

$$x = -\frac{b}{2a}$$

In the quadratic equation in Example 2, $a = -2$, $b = 4$, and $c = 6$. Substituting these values in the formula for the axis of symmetry gives

$$x = -\frac{b}{2a} = -\frac{4}{2(-2)} = -\frac{4}{-4} = 1$$

Thus, the graph is symmetric about the line $x = 1$, and the x-coordinate of the vertex is 1.

The y-coordinate of the vertex can be found by substituting the value of the x-coordinate of the vertex into the quadratic equation and solving for y.

$$\begin{aligned} y &= -2x^2 + 4x + 6 \\ &= -2(1)^2 + 4(1) + 6 \\ &= -2(1) + 4 + 6 \\ &= -2 + 4 + 6 \\ &= 8 \end{aligned}$$

The vertex is at the point $(1, 8)$.

For a quadratic equation of the form $y = ax^2 + bx + c$, the y-coordinate of the vertex can also be found by using the following formula.

y-Coordinate of the Vertex

$$y = \frac{4ac - b^2}{4a}$$

In Example 2,

$$\begin{aligned} y &= \frac{4ac - b^2}{4a} \\ &= \frac{4(-2)(6) - 4^2}{4(-2)} \\ &= \frac{-48 - 16}{-8} = \frac{-64}{-8} = 8 \end{aligned}$$

You may use the method of your choice to find the y-coordinate of the vertex. Both methods result in the same value of y.

3 Use Symmetry to Graph Quadratic Equations

When graphing parabolas, first determine the axis of symmetry and the vertex of the graph. Then select nearby values of x on either side of the axis of symmetry. When plotting points, make use of the symmetry of the graph.

EXAMPLE 3

a) Find the axis of symmetry of the graph of the equation $y = x^2 + 6x + 5$.

b) Find the vertex of the graph.

c) Graph the equation.

Solution

a) $a = 1, \quad b = 6, \quad c = 5.$

$$x = -\frac{b}{2a} = -\frac{6}{2(1)} = -3$$

The parabola is symmetric about the line $x = -3$. The x-coordinate of the vertex is -3.

b) Now find the y-coordinate of the vertex. Substitute -3 for x in the quadratic equation.

$$y = x^2 + 6x + 5$$
$$y = (-3)^2 + 6(-3) + 5 = 9 - 18 + 5 = -4$$

The vertex is at the point $(-3, -4)$.

c) Since the axis of symmetry is $x = -3$, we will select values for x that are greater than -3. It is often helpful to plot each point as it is determined. If a point does not appear to lie on the parabola, check it.

$$y = x^2 + 6x + 5$$

Let $x = -2$, $\quad y = (-2)^2 + 6(-2) + 5 = -3$

Let $x = -1$, $\quad y = (-1)^2 + 6(-1) + 5 = 0$

Let $x = 0$, $\quad y = (0)^2 + 6(0) + 5 = 5$

x	y
-2	-3
-1	0
0	5

These points are plotted in **Figure 10.7a**. Note how we use symmetry to complete the graph in **Figure 10.7b**. The points $(-2, -3)$ and $(-4, -3)$ are each 1 horizontal unit from the axis of symmetry, $x = -3$. The points $(-1, 0)$ and $(-5, 0)$ are each 2 horizontal units from the axis of symmetry, and the points $(0, 5)$ and $(-6, 5)$ are each 3 horizontal units from the axis of symmetry.

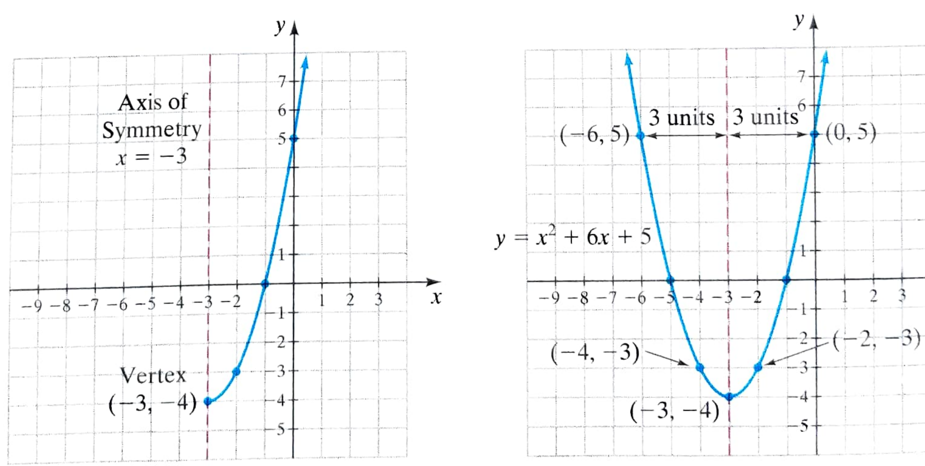


FIGURE 10.7 (a) (b)

Now Try Exercise 25

EXAMPLE 4 Graph $y = -2x^2 + 5x - 4$.

Solution $\quad a = -2, \quad b = 5, \quad c = -4.$

Since $a < 0$, this parabola will open downward.

$$\text{Axis of symmetry: } x = -\frac{b}{2a}$$
$$= -\frac{5}{2(-2)} = -\frac{5}{-4} = \frac{5}{4} \quad \left(\text{or } 1\frac{1}{4}\right)$$

Since the x-value of the vertex is a fraction, we would need to add fractions if we wished to find y by substituting $\frac{5}{4}$ for x in the given equation. Therefore, we will use the formula to find the y-coordinate of the vertex.

$$y = \frac{4ac - b^2}{4a}$$

$$= \frac{4(-2)(-4) - 5^2}{4(-2)} = \frac{32 - 25}{-8} = \frac{7}{-8} = -\frac{7}{8}$$

The vertex of this graph is at the point $\left(\frac{5}{4}, -\frac{7}{8}\right)$. Since the axis of symmetry is $x = \frac{5}{4}$, we will begin by selecting values of x that are greater than $\frac{5}{4}$, or $1\frac{1}{4}$.

$y = -2x^2 + 5x - 4$

Let $x = 2$, $y = -2(2)^2 + 5(2) - 4 = -2$

Let $x = 3$, $y = -2(3)^2 + 5(3) - 4 = -7$

Let $x = 4$, $y = -2(4)^2 + 5(4) - 4 = -16$

x	y
2	−2
3	−7
4	−16

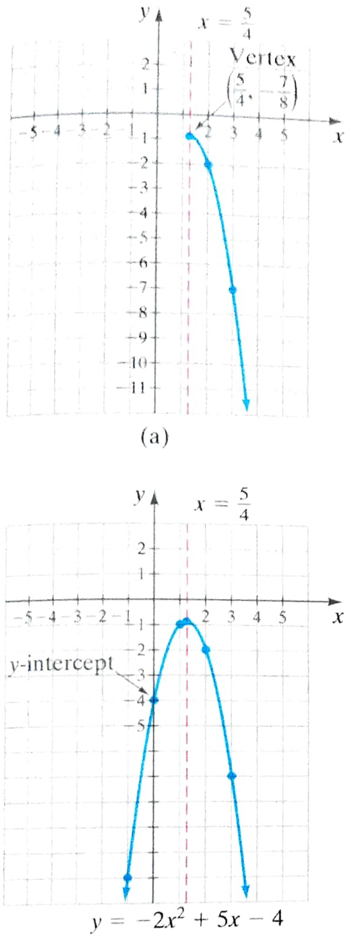


FIGURE 10.8

When the axis of symmetry is a fractional value, be very careful when constructing the graph. You should plot as many additional points as needed. Following we determine some values of y when x is less than $\frac{5}{4}$.

Let $x = 1$ $y = -2(1)^2 + 5(1) - 4 = -1$

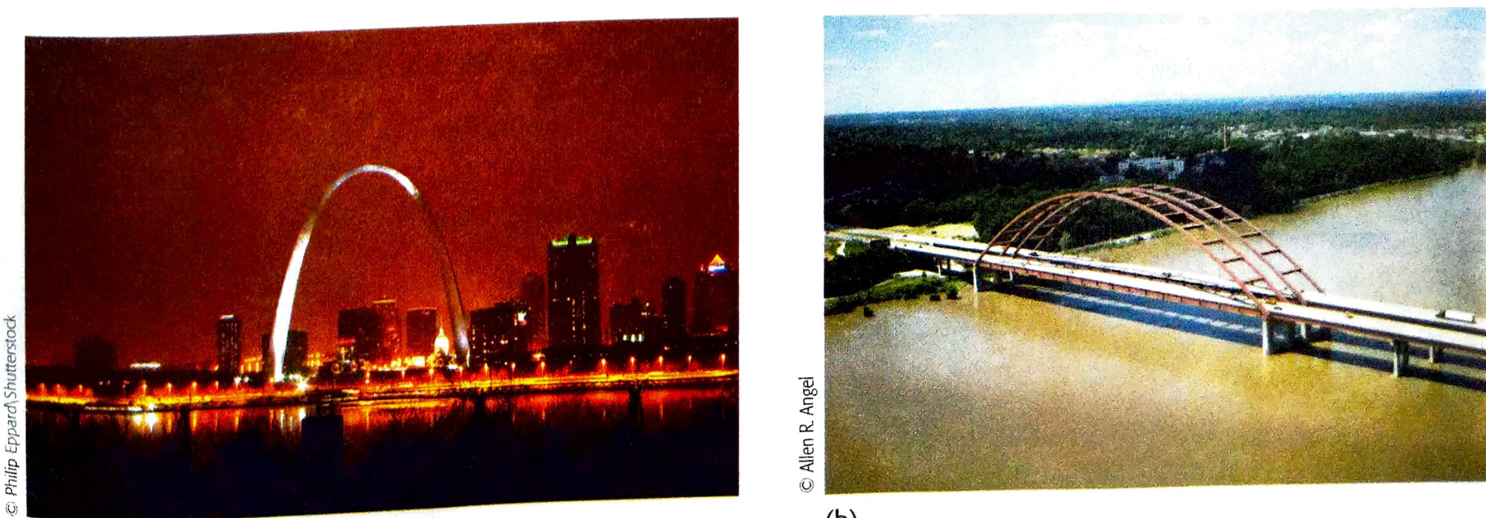
Let $x = 0$ $y = -2(0)^2 + 5(0) - 4 = -4$

Let $x = -1$ $y = -2(-1)^2 + 5(-1) - 4 = -11$

x	y
1	−1
0	−4
−1	−11

Figure 10.8a shows the points plotted on the right side of the axis of symmetry. **Figure 10.8b** shows the completed graph. The point $(4, -16)$ is not shown on the graphs.

Now Try Exercise 19



(a) (b)

FIGURE 10.9 Not every shape that resembles a parabola is a parabola. For example, the St. Louis Arch, **Figure 10.9a** resembles a parabola, but it is not a parabola. However, the bridge over the Mississippi near Jefferson Barracks, Missouri, which connects Missouri and Illinois, **Figure 10.9b**, is a parabola.

4 Find the Intercepts of the Graph of a Quadratic Equation

When graphing parabolas, knowing the location of the intercepts is very helpful. We include finding the intercepts as a part of a general strategy for graphing quadratic equations.

x-intercepts, y-intercepts

An x-**intercept** is a point where a graph crosses the x-axis. An x-intercept will always have the form $(x, 0)$.

A y-**intercept** is a point where a graph crosses the y-axis. A y-intercept will always have the form $(0, y)$.

Understanding Algebra

- To find the x-intercept(s), if they exist, set $y = 0$ and solve for x.
- To find the y-intercept, set $x = 0$ and solve for y.

To find the y-intercept, set $x = 0$ and solve for y. Notice that if we set $x = 0$ in the equation $y = ax^2 + bx + c$, we get $y = a(0)^2 + b(0) + c = 0 + 0 + c$. Thus, *a quadratic equation in the form $y = ax^2 + bx + c$ will always have y-intercept* $(0, c)$.

To find the x-intercept(s), if they exist, set $y = 0$ and solve for x. If we set $y = 0$ in the equation $y = ax^2 + bx + c$ we get $0 = ax^2 + bx + c$ or $ax^2 + bx + c = 0$. To solve this equation we can use one of three methods:

Method 1: Factoring, as explained in Section 5.6

Method 2: Completing the square, as explained in Section 10.2

Method 3: The quadratic formula, as explained in Section 10.3

The number of x-intercepts that a parabola has can be determined by the discriminant, $b^2 - 4ac$.

A quadratic equation of the form $y = ax^2 + bx + c$ will have either two distinct x-intercepts (**Fig. 10.10a**), one x-intercept (**Fig. 10.10b**), or no x-intercept (**Fig. 10.10c**).

Understanding Algebra

For the equation $y = ax^2 + bx + c = 0$, when the discriminant, $b^2 - 4ac$, is

- *positive*: there are two distinct x-intercepts.
- *equal to zero*: there is one x-intercept.
- *negative*: there is no x-intercept.

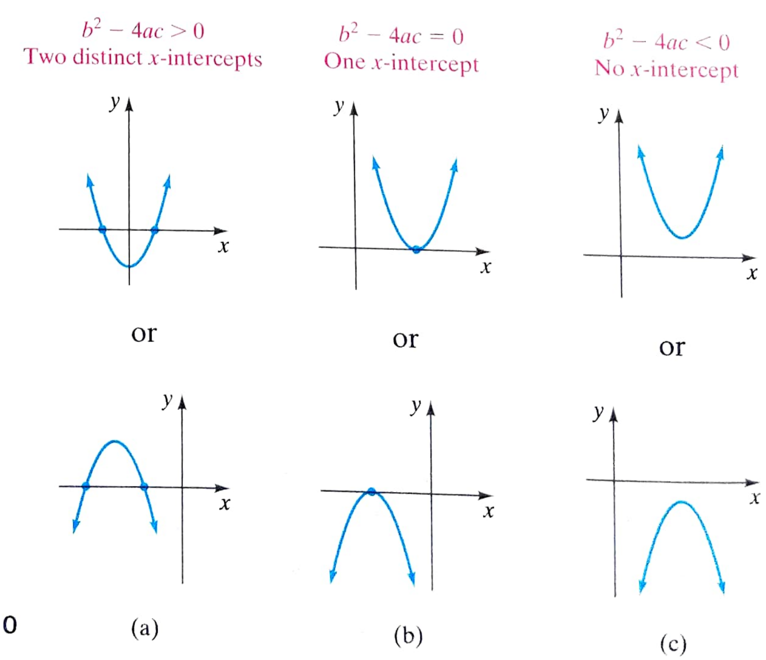


FIGURE 10.10

The x-intercepts can be found algebraically by setting y equal to 0 and solving the resulting equation for x, as we will show in Example 5.

EXAMPLE 5

a) Find the y-intercept of the graph of the equation $y = x^2 - 6x - 7$.

b) Find the x-intercepts of the graph of the equation $y = x^2 - 6x - 7$ by factoring, by completing the square, and by the quadratic formula.

c) Graph the equation.

Solution

a) To find the y-intercept we set $x = 0$ to get

$$y = (0)^2 - 6(0) - 7 = 0 - 0 - 7 = -7.$$

Thus the y-intercept is $(0, -7)$. We could also have noted that the constant term c was -7, therefore the y-intercept is $(0, -7)$.

b) To find the x-intercepts we set y equal to 0 and solve the resulting equation, $x^2 - 6x - 7 = 0$. We will solve this equation by all three algebraic methods.

Method 1: Factoring.

$$\begin{aligned} x^2 - 6x - 7 &= 0 \\ (x - 7)(x + 1) &= 0 \\ x - 7 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = 7 \quad & \quad\quad x = -1 \end{aligned}$$

Method 2: Completing the square.

$$\begin{aligned} x^2 - 6x - 7 &= 0 \\ x^2 - 6x &= 7 \\ x^2 - 6x + 9 &= 7 + 9 \\ (x - 3)^2 &= 16 \\ x - 3 &= \pm 4 \\ x &= 3 \pm 4 \\ x = 3 + 4 \quad &\text{or} \quad x = 3 - 4 \\ x = 7 \quad & \quad\quad x = -1 \end{aligned}$$

Method 3: Quadratic formula.

$$x^2 - 6x - 7 = 0$$

$$a = 1, \qquad b = -6, \qquad c = -7$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-7)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 + 28}}{2} \\ &= \frac{6 \pm \sqrt{64}}{2} \\ &= \frac{6 \pm 8}{2} \end{aligned}$$

$$x = \frac{6 + 8}{2} \quad \text{or} \quad x = \frac{6 - 8}{2}$$

$$= \frac{14}{2} = 7 \qquad\qquad = \frac{-2}{2} = -1$$

Note that the same solutions, 7 and -1, were obtained by all three methods. The graph of the equation $y = x^2 - 6x - 7$ will cross the x-axis at 7 and -1. The x-intercepts are $(7, 0)$ and $(-1, 0)$.

c) Since $a > 0$, this parabola opens upward.

$$\text{axis of symmetry:} \quad x = -\frac{b}{2a} = -\frac{-6}{2(1)} = \frac{6}{2} = 3$$

	$y = x^2 - 6x - 7$
Let $x = 3$,	$y = 3^2 - 6(3) - 7 = -16$
Let $x = 4$,	$y = 4^2 - 6(4) - 7 = -15$
Let $x = 5$,	$y = 5^2 - 6(5) - 7 = -12$
Let $x = 6$,	$y = 6^2 - 6(6) - 7 = -7$
Let $x = 7$,	$y = 7^2 - 6(7) - 7 = 0$
Let $x = 8$,	$y = 8^2 - 6(8) - 7 = 9$

x	y
3	−16
4	−15
5	−12
6	−7
7	0
8	9

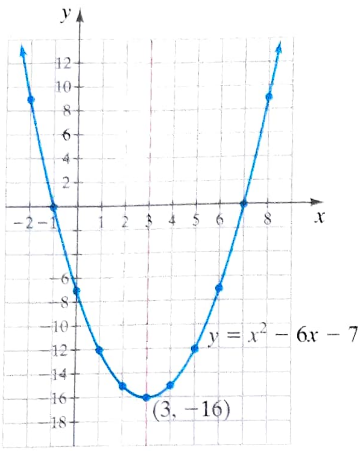


FIGURE 10.11

The vertex is at $(3, -16)$. Again we use symmetry to complete the graph (**Fig. 10.11**). The y-intercept is $(0, -7)$ and the x-intercepts are $(7, 0)$ and $(-1, 0)$. This agrees with the answer obtained in parts **a)** and **b)**.

Now Try Exercise 31

Our discussion of graphing quadratic equations can be summarized with the following guidelines.

Graphing Quadratic Equations of the Form $y = ax^2 + bx + c, a \neq 0$

1. If necessary, rewrite the equation in the form $y = ax^2 + bx + c$.
2. Examine the numerical coefficient a. If a is
   - Positive, then the parabola will open upward.
   - Negative, then the parabola will open downward.
3. Determine the equation of the axis of symmetry using $x = -\frac{b}{2a}$.
4. Determine the vertex of the parabola.
   - The x-coordinate of the vertex is the value determined in step 3.
   - The y-coordinate can be determined by either substituting the x-coordinate into the original equation, or by using the formula $y = \frac{4ac - b^2}{4a}$.
5. Determine the intercepts of the graph.
   - The y-intercept is $(0, c)$ where c is the constant term.
   - The x-intercepts are determined by setting $y = 0$ and solving for x by factoring, completing the square, or using the quadratic formula.
6. Begin your graph by graphing the axis of symmetry, vertex, and intercepts.
7. Select values of x and substitute them into the original equation to find the corresponding y-coordinates. Plot these points on your graph.
8. Use symmetry to plot more points on your graph.
9. Connect the points with a smooth curve.

EXERCISE SET 10.4

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Warm-Up Exercises

Fill in the blanks with the appropriate word, phrase, or symbol(s) from the following list.

two	downward	one	axis of symmetry	parabola
no	y-coordinate	upward	vertex	x-coordinate

1. The graph of $y = ax^2 + bx + c$ is a ________.

2. The graph of $y = 3x^2 - \frac{1}{2}x + 7$ opens ________.

3. The graph of $y = -\frac{1}{4}x^2 + x - 4$ opens ________.

4. The lowest point on a parabola that opens upward is the ________.

5. The formula $x = -\frac{b}{2a}$ is used to find the ________ of the vertex.

6. The formula $y = \frac{4ac - b^2}{4a}$ is used to find the ________ of the vertex.

7. The graph of a parabola is symmetric about the ________.

8. If $b^2 - 4ac = 0$, the graph of the quadratic equation $y = ax^2 + bx + c$ has ________ x-intercept(s).

Practice the Skills

Indicate the axis of symmetry, the coordinates of the vertex, and whether the parabola opens upward or downward.

9. $y = x^2 + 2x - 7$
10. $y = x^2 + 4x - 3$
11. $y = 4x^2 + 8x + 3$
12. $y = -x^2 + 8x - 11$
13. $y = -3x^2 + 2x + 1$
14. $y = x^2 + 3x - 5$
15. $y = -x^2 + 3x - 4$
16. $y = 3x^2 - 2x + 6$
17. $y = 2x^2 + 3x + 2$
18. $y = -2x^2 - 6x - 1$
19. $y = -x^2 + x + 8$
20. $y = -5x^2 + 6x - 3$

Graph each quadratic equation and determine the x-intercepts, if they exist.

21. $y = x^2 + 3$
22. $y = -x^2 + 9$
23. $y = -x^2 + 5$
24. $y = x^2 - 1$
25. $y = x^2 + 4x + 3$
26. $y = 2x^2 + 1$
27. $y = x^2 + 4x + 4$
28. $y = x^2 - 4x + 4$
29. $y = -x^2 - 5x - 4$
30. $y = -x^2 - 5x + 6$
31. $y = x^2 + 5x - 6$
32. $y = x^2 - 4x - 5$
33. $y = x^2 + 5x - 14$
34. $y = 2x^2 + 4x + 2$
35. $y = x^2 - 6x + 9$
36. $y = x^2 + 8x + 16$
37. $y = x^2 - 6x$
38. $y = -x^2 + 5x$
39. $y = x^2 - 2x + 1$
40. $y = x^2 - x + 1$
41. $y = -x^2 + 7x - 10$
42. $y = -x^2 + 5x - 6$
43. $y = 4x^2 + 12x + 9$
44. $y = 2x^2 + 3x - 2$
45. $y = -2x^2 + 3x - 2$
46. $y = -4x^2 - 6x + 4$
47. $y = 2x^2 - x - 15$
48. $y = x^2 - 5x + 4$

Using the discriminant, determine the number of x-intercepts the graph of each equation will have. Do not graph the equation.

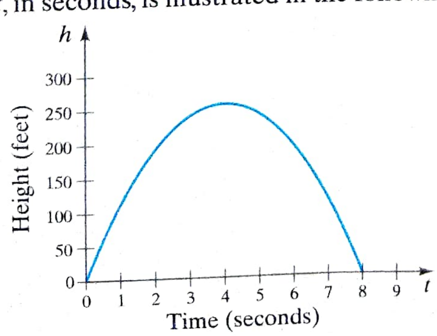
49. $y = 4x^2 - 2x - 7$
50. $y = -x^2 - 3$
51. $y = 4x^2 - 6x - 5$
52. $y = x^2 - 6x + 9$
53. $y = x^2 - 22x + 121$
54. $y = -4.3x^2 + 2.4x$
55. $y = 5.7x^2 + 2x - 1.5$
56. $y = 5x^2 - 7.1x + 6.3$

Problem Solving

The graph of a quadratic equation of the form $y = ax^2 + bx + c$ is a parabola. The value of a (the coefficient of the squared term in the equation) and the vertex of the parabola are given. Determine the number of x-intercepts the parabola will have. Explain how you determined your answer.

57. $a = -2$, vertex at $(0, -4)$
58. $a = 2$, vertex at $(1, -3)$
59. $a = -3$, vertex at $(-5, 0)$
60. $a = -1$, vertex at $(2, -7)$

61. **Height above Ground** An object is projected upward from the ground. The height of the object above the ground, in feet, at time t, in seconds, is illustrated in the following graph.



a) Estimate the maximum height the object will obtain.

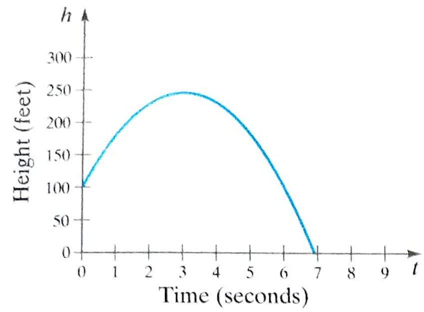
b) How long will it take for the object to reach its maximum height?

c) Estimate how long it will take for the object to strike the ground.

d) Estimate the object's height at 2 seconds and at 5 seconds.

62. **Height above Ground** A ball is projected upward from the top of a building that is 100 feet above the ground. The

height of the ball above the ground, in feet, at time t, in seconds, is illustrated in the following graph.



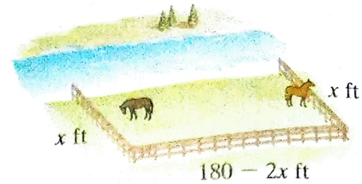
a) Estimate the maximum height the ball will obtain.

b) How long will it take for the ball to reach its maximum height?

c) Approximately how long will it take for the ball to strike the ground?

d) Estimate the ball's height above the ground at 2 seconds and at 5 seconds.

63. Maximum Area An area is to be fenced in along a river as shown in the figure on the right. Only 180 feet of fencing is available. The fenced-in area is $A = \text{length} \cdot \text{width}$ or $A = x(180 - 2x) = -2x^2 + 180x$.



a) Graph $A = -2x^2 + 180x$.

b) Using the graph, estimate the value of x that will yield the maximum area.

c) Estimate the maximum area.

64. Maximum Area See Exercise 63. If 260 feet of fencing is available, the fenced-in area is $A = x(260 - 2x) = -2x^2 + 260x$.

a) Graph $A = -2x^2 + 260x$.

b) Using the graph, estimate the value of x that yields the maximum area.

c) Estimate the maximum area.

65. Will the equations below have the same x-intercepts when graphed? Explain how you determined your answer.

$$y = x^2 - 2x - 15 \quad \text{and} \quad y = -x^2 + 2x + 15$$

66. a) How will the graphs of the following equations compare? Explain how you determined your answer.

$$y = x^2 - 2x - 8 \quad \text{and} \quad y = -x^2 + 2x + 8$$

b) Graph $y = x^2 - 2x - 8$ and $y = -x^2 + 2x + 8$ on the same axes.

Concept/Writing Exercises

67. Explain how to find the coordinates of the vertex of a parabola.

68. What determines whether the graph of a quadratic equation of the form $y = ax^2 + bx + c, a \neq 0$, is a parabola that opens upward or downward? Explain your answer.

69. a) What are the x-intercepts of a graph?

b) How can you find the x-intercepts of a graph algebraically?

70. What does it mean when we say that graphs of quadratic equations of the form $y = ax^2 + bx + c$ have symmetry about the axis of symmetry?

71. When graphing a quadratic equation of the form $y = ax^2 + bx + c$, what is the equation of the axis of symmetry?

72. How many x-intercepts will the graph of a quadratic equation have if the discriminant has a value of

a) 19

b) −5

c) 0

Challenge Problems

73. a) Graph the quadratic equation $y = -x^2 + 6x$.

b) On the same axes, graph the quadratic equation $y = x^2 - 2x$.

c) Estimate the points of intersection of the graphs. The points represent the solution to the system of equations.

74. a) Graph the quadratic equation $y = x^2 + 2x - 3$.

b) On the same axes, graph the quadratic equation $y = -x^2 + 1$.

c) Estimate the points of intersection of the graphs. The points represent the solution to the system of equations.

Cumulative Review Exercises

[6.4] **75.** Subtract $\dfrac{5}{x+3} - \dfrac{x-2}{x-4}$.

[6.6] **76.** Solve $\dfrac{1}{3}(x + 6) = 3 - \dfrac{1}{4}(x - 5)$.

[8.3] **77.** Solve the system of equations.

$$2x + 3y = -3$$
$$3x + 5y = -7$$

[9.5] **78.** Solve $\sqrt{x + 9} - x = -3$.