6.6 Solving Rational Equations

1 Solve rational equations with integer denominators.

2 Solve rational equations where a variable appears in a denominator.

1 Solve Rational Equations with Integer Denominators

A **rational equation** is one that contains one or more rational expressions. A rational equation may contain rational coefficients, such as $\frac{1}{2}x + \frac{3}{5}x = 8$ or $\frac{x}{2} + \frac{3x}{5} = 8$. A rational equation may also have a variable in a denominator, such as $\frac{4}{x-2} = 5$.

The emphasis of this section will be on solving rational equations where a variable appears in a denominator.

To Solve Rational Equations

1. Determine the least common denominator (LCD) of all fractions in the equation.
2. Multiply *both* sides of the equation by the LCD. *This will result in every term in the equation being multiplied by the LCD.*
3. Remove any parentheses and combine like terms on each side of the equation.
4. Solve the equation using the properties discussed in earlier chapters.
5. Check your solution in the *original* equation.

The purpose of multiplying both sides of the equation by the LCD (step 2) is to eliminate all fractions from the equation. After both sides of the equation are multiplied by the LCD, the resulting equation should contain no fractions.

EXAMPLE 1 Solve $\frac{t}{4} - \frac{t}{5} = 1$ for t.

Solution The LCD of 4 and 5 is 20. Multiply both sides of the equation by 20.

$$\frac{t}{4} - \frac{t}{5} = 1$$

$$20\left(\frac{t}{4} - \frac{t}{5}\right) = 20 \cdot 1 \qquad \text{Multiply both sides by the LCD, 20.}$$

$$20\left(\frac{t}{4}\right) - 20\left(\frac{t}{5}\right) = 20 \qquad \text{Distributive property}$$

$$5t - 4t = 20$$

$$t = 20$$

Check

$$\frac{t}{4} - \frac{t}{5} = 1$$

$$\frac{20}{4} - \frac{20}{5} \stackrel{?}{=} 1$$

$$5 - 4 \stackrel{?}{=} 1$$

$$1 = 1 \qquad \text{True}$$

The solution is 20.

Now Try Exercise 13

EXAMPLE 2 Solve $\frac{x-5}{30} = \frac{4}{5} - \frac{x-1}{10}$.

Solution Multiply both sides of the equation by the LCD, 30.

$$\frac{x-5}{30} = \frac{4}{5} - \frac{x-1}{10}$$

$$30\left(\frac{x-5}{30}\right) = 30\left(\frac{4}{5} - \frac{x-1}{10}\right)$$ Multiply both sides by the LCD, 30.

$$x - 5 = 30\left(\frac{4}{5}\right) - 30\left(\frac{x-1}{10}\right)$$ Distributive property

$$x - 5 = 24 - 3(x - 1)$$

$$x - 5 = 24 - 3x + 3$$ Distributive property

$$x - 5 = -3x + 27$$ Combined like terms.

$$4x - 5 = 27$$ $3x$ was added to both sides.

$$4x = 32$$ 5 was added to both sides.

$$x = 8$$ Both sides were divided by 4.

A check will show that the answer is 8.

Now Try Exercise 27

2 Solve Rational Equations Where a Variable Appears in a Denominator

When solving a rational equation where a variable appears in any denominator, you *must* check your answer. *Whenever a variable appears in any denominator of a rational equation, it is necessary to check your answer in the original equation. If the answer obtained makes any denominator equal to zero, that value is not a solution to the equation.* Such values are called **extraneous roots** or **extraneous solutions**.

EXAMPLE 3 Solve $4 - \frac{5}{x} = \frac{3}{2}$.

Solution Multiply both sides of the equation by the LCD, $2x$.

$$2x\left(4 - \frac{5}{x}\right) = \left(\frac{3}{2}\right) \cdot 2x$$ Multiply both sides by the LCD, $2x$.

$$2x(4) - 2x\left(\frac{5}{x}\right) = \left(\frac{3}{2}\right) \cdot 2x$$ Distributive property

$$8x - 10 = 3x$$

$$5x - 10 = 0$$ $3x$ was subtracted from both sides.

$$5x = 10$$ 10 was added to both sides.

$$x = 2$$

Check

$$4 - \frac{5}{x} = \frac{3}{2}$$

$$4 - \frac{5}{2} \stackrel{?}{=} \frac{3}{2}$$

$$\frac{8}{2} - \frac{5}{2} \stackrel{?}{=} \frac{3}{2}$$

$$\frac{3}{2} = \frac{3}{2}$$ True

Since 2 does check, it is the solution to the equation.

Now Try Exercise 17

EXAMPLE 4 Solve $\frac{p-5}{p+3} = \frac{1}{5}$.

Solution The LCD is $5(p+3)$. Multiply both sides of the equation by the LCD.

$$5(p+3) \cdot \frac{(p-5)}{p+3} = \frac{1}{5} \cdot 5(p+3)$$
$$5(p-5) = 1(p+3)$$
$$5p - 25 = p + 3$$
$$4p - 25 = 3$$
$$4p = 28$$
$$p = 7$$

A check will show that 7 is the solution.

Now Try Exercise 43

In Section 2.7 we illustrated that proportions of the form

$$\frac{a}{b} = \frac{c}{d}$$

can be cross-multiplied to obtain $a \cdot d = b \cdot c$. Example 4 is a proportion and can also be solved by cross-multiplying, as we will do in Example 5.

EXAMPLE 5 Use cross-multiplication to solve $\frac{9}{x+1} = \frac{5}{x-3}$.

Solution

$$\frac{9}{x+1} = \frac{5}{x-3}$$
$$9(x-3) = 5(x+1) \quad \text{Cross-multiplied.}$$
$$9x - 27 = 5x + 5 \quad \text{Distributive property was used}$$
$$4x - 27 = 5$$
$$4x = 32$$
$$x = 8$$

A check will show that 8 is the solution to the equation.

Now Try Exercise 4[illegible]

The following examples involve quadratic equations. Recall from Section 5.6 that quadratic equations have the form $ax^2 + bx + c = 0$, where $a \neq 0$.

EXAMPLE 6 Solve $x + \frac{12}{x} = -7$.

Solution

$$x + \frac{12}{x} = -7$$
$$x \cdot \left(x + \frac{12}{x}\right) = -7 \cdot x \quad \text{Multiply both sides by } x.$$
$$x(x) + x\left(\frac{12}{x}\right) = -7x \quad \text{Distributive property was used.}$$
$$x^2 + 12 = -7x$$
$$x^2 + 7x + 12 = 0 \quad 7x \text{ was added to both sides.}$$
$$(x+3)(x+4) = 0 \quad \text{Factored.}$$
$$x + 3 = 0 \quad \text{or} \quad x + 4 = 0 \quad \text{Zero-factor property}$$
$$x = -3 \qquad\qquad x = -4$$

Check

$x = -3$

$$x + \frac{12}{x} = -7$$
$$-3 + \frac{12}{-3} \stackrel{?}{=} -7$$
$$-3 + (-4) \stackrel{?}{=} -7$$
$$-7 = -7 \quad \text{True}$$

$x = -4$

$$x + \frac{12}{x} = -7$$
$$-4 + \frac{12}{-4} \stackrel{?}{=} -7$$
$$-4 + (-3) \stackrel{?}{=} -7$$
$$-7 = -7 \quad \text{True}$$

The solutions are -3 and -4.

Now Try Exercise 57

EXAMPLE 7 Solve $\frac{x^2 - 2x}{x - 6} = \frac{24}{x - 6}$.

Solution If we try to solve this equation using cross-multiplication we will get a cubic equation. We will solve this equation by multiplying both sides of the equation by the LCD, $x - 6$.

Understanding Algebra

When multiplying both sides of an equation by a variable expression, checking answers is crucial to determine if any of your answers are extraneous solutions that must be eliminated from the final answer.

$$\frac{x^2 - 2x}{x - 6} = \frac{24}{x - 6}$$

$$\cancel{x - 6} \cdot \frac{x^2 - 2x}{\cancel{x - 6}} = \frac{24}{\cancel{x - 6}} \cdot \cancel{x - 6} \quad \text{Multiply both sides by the LCD, } x - 6.$$

$$x^2 - 2x = 24$$

$$x^2 - 2x - 24 = 0 \quad \text{24 was subtracted from both sides.}$$

$$(x + 4)(x - 6) = 0 \quad \text{Factored}$$

$$x + 4 = 0 \quad \text{or} \quad x - 6 = 0 \quad \text{Zero-factor property}$$

$$x = -4 \qquad\qquad x = 6$$

Check

$x = -4$

$$\frac{x^2 - 2x}{x - 6} = \frac{24}{x - 6}$$
$$\frac{(-4)^2 - 2(-4)}{-4 - 6} \stackrel{?}{=} \frac{24}{-4 - 6}$$
$$\frac{16 + 8}{-10} \stackrel{?}{=} \frac{24}{-10}$$
$$\frac{24}{-10} = \frac{24}{-10} \quad \text{True}$$

$x = 6$

$$\frac{x^2 - 2x}{x - 6} = \frac{24}{x - 6}$$
$$\frac{6^2 - 2(6)}{6 - 6} \stackrel{?}{=} \frac{24}{6 - 6}$$
$$\frac{24}{0} = \frac{24}{0}$$

↑ ↑

Since the denominator is 0, and we cannot divide by 0, 6 is not a solution.

Since $\frac{24}{0}$ is not a real number, 6 is an extraneous solution. Thus, this equation has only one solution, -4.

Now Try Exercise 47

Helpful Hint

Remember, when solving a rational equation in which a variable appears in a denominator, you must check *all* your answers to make sure that none is an extraneous root. If any of your answers make any denominator 0, that answer is an extraneous root and not a true solution.

EXAMPLE 8 Solve $\frac{5w}{w^2 - 4} + \frac{1}{w - 2} = \frac{4}{w + 2}$.

Solution First factor $w^2 - 4$.

$$\frac{5w}{(w + 2)(w - 2)} + \frac{1}{w - 2} = \frac{4}{w + 2}$$

Multiply both sides of the equation by the LCD, $(w + 2)(w - 2)$.

$$(w + 2)(w - 2)\left[\frac{5w}{(w + 2)(w - 2)} + \frac{1}{w - 2}\right] = \frac{4}{w + 2}\cdot(w + 2)(w - 2)$$

$$(w + 2)(w - 2)\cdot\frac{5w}{(w + 2)(w - 2)} + (w + 2)(w - 2)\cdot\frac{1}{w - 2} = \frac{4}{w + 2}\cdot(w + 2)(w - 2)$$

$$\cancel{(w + 2)}\cancel{(w - 2)}\cdot\frac{5w}{\cancel{(w + 2)}\cancel{(w - 2)}} + (w + 2)\cancel{(w - 2)}\cdot\frac{1}{\cancel{w - 2}} = \frac{4}{\cancel{w + 2}}\cdot\cancel{(w + 2)}(w - 2)$$

$$5w + (w + 2) = 4(w - 2)$$

$$6w + 2 = 4w - 8$$

$$2w + 2 = -8$$

$$2w = -10$$

$$w = -5$$

A check will show that -5 is the solution to the equation.

Now Try Exercise 65

Helpful Hint

Some students confuse adding and subtracting rational expressions with solving rational equations. When adding or subtracting rational expressions, we must rewrite each expression with a common denominator. When solving a rational equation, we multiply both sides of the equation by the LCD to eliminate fractions from the equation. Consider the following two problems. Note that the one on the right is an equation because it contains an equals sign. We will work both problems. The LCD for both problems is $x(x + 4)$.

Adding Rational Expressions

$$\frac{x + 2}{x + 4} + \frac{3}{x}$$

We rewrite each fraction with the LCD, $x(x + 4)$.

$$= \frac{x}{x}\cdot\frac{x + 2}{x + 4} + \frac{3}{x}\cdot\frac{x + 4}{x + 4}$$

$$= \frac{x(x + 2)}{x(x + 4)} + \frac{3(x + 4)}{x(x + 4)}$$

$$= \frac{x^2 + 2x}{x(x + 4)} + \frac{3x + 12}{x(x + 4)}$$

$$= \frac{x^2 + 2x + 3x + 12}{x(x + 4)}$$

$$= \frac{x^2 + 5x + 12}{x(x + 4)}$$

Solving Rational Equations

$$\frac{x + 2}{x + 4} = \frac{3}{x}$$

We eliminate fractions by multiplying both sides of the equation by the LCD, $x(x + 4)$.

$$(x)\cancel{(x + 4)}\left(\frac{x + 2}{\cancel{x + 4}}\right) = \frac{3}{\cancel{x}}(\cancel{x})(x + 4)$$

$$x(x + 2) = 3(x + 4)$$

$$x^2 + 2x = 3x + 12$$

$$x^2 - x - 12 = 0$$

$$(x - 4)(x + 3) = 0$$

$$x - 4 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 4 \qquad\qquad x = -3$$

The numbers 4 and -3 on the right will both check and are thus solutions to the equation.

Understanding Algebra

When adding and subtracting rational expressions, we usually end up with an algebraic expression.

When solving rational equations, the solution, if one exists, will be a numerical value or values.

EXERCISE SET 6.6

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Warm-Up Exercises

Fill in the blanks with the appropriate word, phrase, or symbol(s) from the following list.

LCD	check	rational expression	No	$(x+1)(x-3)$
rational equation	multiplied	$x+1$	Yes	

1. When solving rational equations, both sides of the equation are __________ by the LCD.

2. When solving rational equations with variable denominators, it is very important to __________ your answers.

3. $\frac{x}{2} - \frac{x}{3} + \frac{5}{2x+7}$ is an example of a __________.

4. The first step in solving a rational equation is determining the __________ of all fractions.

5. $\frac{x}{2} - \frac{x}{3} = \frac{5}{2x+7}$ is an example of a __________.

6. (Yes or No) Is $x = 2$ a solution to the equation $\frac{3}{x-2} + \frac{2}{x+2} = \frac{4x}{x^2-4}$? __________

7. (Yes or No) Is $x = 1$ a solution to the equation $\frac{3}{x-2} + \frac{2}{x+2} = \frac{7x}{x^2-4}$? __________

8. The first step in solving the equation $\frac{9}{x+1} = \frac{5}{x-3}$ is to multiply both sides of the equation by the algebraic expression __________.

Practice the Skills

Solve each equation and check your solution. See Examples 1 and 2.

9. $\frac{x}{3} + \frac{x}{2} = 10$

10. $\frac{x}{3} - \frac{x}{2} = 10$

11. $\frac{y}{6} - \frac{y}{4} = \frac{1}{2}$

12. $\frac{x}{4} - \frac{x}{6} = \frac{1}{2}$

13. $\frac{x}{3} - \frac{x}{4} = 1$

14. $\frac{t}{5} - \frac{t}{6} = 2$

15. $\frac{r}{6} = \frac{r}{4} + \frac{1}{3}$

16. $\frac{n}{5} = \frac{n}{6} + \frac{2}{3}$

17. $\frac{z}{2} + 6 = \frac{z}{5}$

18. $\frac{3w}{5} - 6 = w$

19. $\frac{z}{6} + \frac{2}{3} = \frac{z}{5} - \frac{1}{3}$

20. $\frac{m-2}{6} = \frac{2}{3} + \frac{m}{12}$

21. $d + 7 = \frac{3}{2}d + 5$

22. $\frac{q}{5} + \frac{q}{2} = \frac{21}{10}$

23. $3k + \frac{1}{6} = 4k - 4$

24. $\frac{p}{4} + \frac{1}{4} = \frac{p}{3} - \frac{1}{2}$

25. $\frac{n+6}{3} = \frac{5(n-8)}{10}$

26. $\frac{3(x-6)}{5} = \frac{4(x+2)}{8}$

27. $\frac{x-5}{15} = \frac{3}{5} - \frac{x-4}{10}$

28. $\frac{z+4}{6} = \frac{3}{2} - \frac{2z+2}{12}$

29. $\frac{-p+1}{4} + \frac{13}{20} = \frac{p}{5} - \frac{p-1}{2}$

30. $\frac{1}{10} - \frac{n+1}{6} = \frac{1}{5} - \frac{n+10}{15}$

31. $\frac{d-3}{4} + \frac{1}{15} = \frac{2d+1}{3} - \frac{34}{15}$

32. $\frac{t+4}{5} = \frac{5}{8} + \frac{t+7}{40}$

Solve each equation and check your solution. See Examples 3–8.

33. $2 + \frac{3}{x} = \frac{11}{4}$

34. $3 - \frac{1}{x} = \frac{14}{5}$

35. $7 - \frac{5}{x} = \frac{9}{2}$

36. $4 + \frac{3}{z} = \frac{9}{2}$

37. $\frac{4}{n} - \frac{3}{2n} = \frac{1}{2}$

38. $\frac{5}{3x} + \frac{2}{x} = 1$

39. $\frac{x-1}{x-5} = \frac{4}{x-5}$

40. $\frac{2x+3}{x+2} = \frac{3}{2}$

41. $\frac{5}{a+3} = \frac{4}{a+1}$

42. $\frac{5}{x+2} = \frac{1}{x-4}$

43. $\frac{y+3}{y-3} = \frac{6}{4}$

44. $\frac{x}{x+6} = \frac{2}{5}$

45. $\frac{2x-3}{x-4} = \frac{5}{x-4}$

46. $\frac{3}{x} + 9 = \frac{3}{x}$

47. $\frac{x^2}{x-3} = \frac{9}{x-3}$

48. $\frac{x^2}{x+5} = \frac{25}{x+5}$

49. $\frac{n-3}{n+2} = \frac{n+4}{n+10}$

50. $\frac{x+5}{x+1} = \frac{x-6}{x-3}$

51. $\frac{1}{r} = \frac{3r}{8r+3}$

52. $\frac{1}{r} = \frac{2r}{r+15}$

53. $\frac{k}{k+2} = \frac{3}{k-2}$

54. $\frac{3a-2}{2a+2} = \frac{3}{a-1}$

55. $\frac{4}{r} + r = \frac{20}{r}$

56. $a + \frac{5}{a} = \frac{14}{a}$

57. $x + \frac{20}{x} = -9$

58. $x - \frac{32}{x} = 4$

59. $\frac{3y-2}{y+1} = 4 - \frac{y+2}{y-1}$

60. $\frac{2b}{b+1} = 2 - \frac{5}{2b}$

61. $\frac{1}{x+3} + \frac{1}{x-3} = \frac{-5}{x^2-9}$

62. $\frac{t+2}{t-5} - \frac{3}{4} = \frac{6}{t-5}$

63. $\frac{x}{x-3} + \frac{3}{2} = \frac{3}{x-3}$

64. $\frac{y}{2y+2} + \frac{2y-16}{4y+4} = \frac{y-3}{y+1}$

65. $\frac{3}{x-5} - \frac{4}{x+5} = \frac{11}{x^2-25}$

66. $\frac{2n^2-15}{n^2+n-6} = \frac{n+1}{n+3} + \frac{n-3}{n-2}$

67. $\frac{3x}{x^2-9}+\frac{1}{x-3}=\frac{3}{x+3}$

68. $\frac{3}{x+3}+\frac{5}{x+4}=\frac{12x+7}{x^2+7x+12}$

69. $\frac{1}{y-1}+\frac{1}{2}=\frac{2}{y^2-1}$

70. $\frac{2y}{y+2}=\frac{y}{y+3}-\frac{3}{y^2+5y+6}$

71. $\frac{3t}{6t+6}+\frac{t}{2t+2}=\frac{2t-3}{t+1}$

72. $\frac{2}{x-2}-\frac{1}{x+1}=\frac{2}{x^2-x-2}$

Problem Solving

In Exercises 73–78, determine the solution by observation. Explain how you determined your answer.

73. $\frac{3}{x-2}=\frac{x-2}{x-2}$

74. $\frac{1}{2}+\frac{x}{2}=\frac{5}{2}$

75. $\frac{x}{x-6}+\frac{x}{x-6}=0$

76. $\frac{x}{4}+\frac{3x}{4}=x$

77. $\frac{x-2}{3}+\frac{x-2}{3}=\frac{2x-4}{3}$

78. $\frac{3}{x}-\frac{1}{x}=\frac{2}{x}$

79. Optics A formula frequently used in optics is

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$$

where p represents the distance of the object from a mirror (or lens), q represents the distance of the image from the mirror (or lens), and f represents the focal length of the mirror (or lens). If a mirror has a focal length of 10 centimeters, how far from the mirror will the image appear when the object is 30 centimeters from the mirror?

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Challenge Problems

80. a) Explain why the equation $\frac{x^2}{x-3}=\frac{9}{x-3}$ cannot be solved by cross-multiplying using the material presented in the book.

b) Solve the equation given in part **a)**.

81. Solve the equation $\frac{x-4}{x^2-2x}=\frac{-4}{x^2-4}$.

82. Electrical Resistance In electronics the total resistance R_T, of resistors wired in a parallel circuit is determined by the formula

$$\frac{1}{R_T}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\cdots+\frac{1}{R_n}$$

where $R_1, R_2, R_3, \ldots, R_n$ are the resistances of the individual resistors (measured in ohms) in the circuit.

a) Find the total resistance if two resistors, one of 200 ohms and the other of 300 ohms, are wired in a parallel circuit.

b) If three identical resistors are to be wired in parallel, what should be the resistance of each resistor if the total resistance of the circuit is to be 300 ohms?

83. Can an equation of the form $\frac{a}{x}+1=\frac{a}{x}$ have a real number solution for any real number a? Explain your answer.

Group Activity

Discuss and answer Exercise 84 as a group.

84. a) As a group, discuss two different methods you can use to solve the equation $\frac{x+3}{5}=\frac{x}{4}$.

b) Group member 1: Solve the equation by obtaining a common denominator.

Group member 2: Solve the equation by cross-multiplying.

Group member 3: Check the results of group member 1 and group member 2.

c) Individually, create another equation by taking the reciprocal of each term in the equation in part **a)**. Compare your results. Do you think that the reciprocal of the answer you found in part **b)** will be the solution to this equation? Explain.

d) Individually, solve the equation you found in part **c)** and check your answer. Compare your work with the other group members. Was the conclusion you came to in part **c)** correct? Explain.

e) As a group, solve the equation $\frac{1}{x}+\frac{1}{3}=\frac{2}{x}$. Check your result.

f) As a group, create another equation by taking the reciprocal of each term of the equation in part **e)**. Do you think that the reciprocal of the answer you found in part **e)** will be the solution to this equation? Explain.

g) Individually, solve the equation you found in part **f)** and check your answer. Compare your work with the other group members. Did your group make the correct conclusion in part **f)**? Explain.

h) As a group, discuss the relationship between the solution to the equation $\frac{7}{x-9}=\frac{3}{x}$ and the solution to the equation $\frac{x-9}{7}=\frac{x}{3}$. Explain your answer.